SOLUTIONS OF THE CUBIC FERMAT EQUATION IN QUADRATIC FIELDS

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Abstract. We give necessary and sufficient conditions on a squarefree integer d for there to be non-trivial solutions to $x^3 + y^3 = z^3$ in $\mathbb{Q}(\sqrt{d})$, conditional on the Birch and Swinnerton-Dyer conjecture. These conditions are similar to those obtained by J. Tunnell in his solution to the congruent number problem.

1. Introduction and Statement of Results

The enigmatic claim of Fermat that the equation

$$x^n + y^n = z^n$$

has only the trivial solutions (those with at least one of x, y and z zero) in integers when $n \geq 3$ has to a large extent shaped the development of number theory over the course of the last three hundred years. These developments culminated in the theory used by Andrew Wiles in [28] to finally justify Fermat's claim.

In light of Fermat's claim and Wiles's proof, it is natural to ask the following question: for which fields K does the equation $x^n + y^n = z^n$ have a non-trivial solution in K? Two notable results on this question are the following. In [16], it is shown that the equation $x^n + y^n = z^n$ has no non-trivial solutions in $\mathbb{Q}(\sqrt{2})$ provided $n \geq 4$. Their proof uses similar ingredients to Wiles's work.

In [10], Debarre and Klassen use Faltings's work on the rational points on subvarieties of abelian varieties to prove that for $n \geq 3$ and $n \neq 6$, the equation $x^n + y^n = z^n$ has only finitely many solutions (x, y, z) where the variables belong to any number field K with $[K : \mathbb{Q}] \leq n - 2$. Indeed, the work of Aigner shows that when $n = 4$ the only non-trivial solution to $x^n + y^n = z^n$ with x, y and z in any quadratic field is

$$\left(\frac{1+\sqrt{-7}}{2}\right)^4 + \left(\frac{1-\sqrt{-7}}{2}\right)^4 = 1^4,$$

and when $n = 6$ or $n = 9$, there are no non-trivial solutions in quadratic fields.

We now turn to the problem of solutions to $x^3 + y^3 = z^3$ in quadratic fields $\mathbb{Q}(\sqrt{d})$. For some choices of d there are solutions, such as

$$(18 + 17\sqrt{2})^3 + (18 - 17\sqrt{2})^3 = 42^3$$

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for $d = 2$, while for other choices (such as $d = 3$) there are no non-trivial solutions. In 1913, Fueter [11] showed that if $d < 0$ and $d \equiv 2 \pmod 3$, then there are no solutions if 3 does not divide the class number of $\mathbb{Q}(\sqrt{d})$. Fueter also proved in [12] that there is a non-trivial solution to $x^3 + y^3 = z^3$ in $\mathbb{Q}(\sqrt{d})$ if and only if there is one in $\mathbb{Q}(\sqrt{-3d})$.

In 1915, Burnside [8] showed that every solution to $x^3+y^3 = z^3$ in a quadratic field takes the form

$$\begin{aligned} x &= -3 + \sqrt{-3(1+4k^3)}, \\ y &= -3 - \sqrt{-3(1+4k^3)}, \text{ and} \\ z &= 6k \end{aligned}$$

up to scaling. Here k is any rational number not equal to 0 or -1. This, however, does not answer the question of whether or not there are solutions in $\mathbb{Q}(\sqrt{d})$ for given d since it is not clear whether

$$dy^2 = -3(1+4k^3)$$

has a solution with k and y both rational.

In a series of papers [1], [2], [3], [4], Aigner considered this problem (see [21], Chapter XIII, Section 10 for a discussion in English). He showed that there are no solutions in $\mathbb{Q}(\sqrt{-3d})$ if $d > 0$, $d \equiv 1 \pmod 3$, and 3 does not divide the class number of $\mathbb{Q}(\sqrt{-3d})$. He also developed general criteria to rule out the existence of a solution. In particular, there are "obstructing integers" k with the property that there are no solutions in $\mathbb{Q}(\sqrt{\pm d})$ if $d = kR$, where R is a product of primes congruent to 1 (mod 3) for which 2 is a cubic non-residue.

The goal of the present paper is to give a complete classification of the fields $\mathbb{Q}(\sqrt{d})$ in which $x^3 + y^3 = z^3$ has a solution. Our main result is the following.

Theorem 1. *Assume the Birch and Swinnerton-Dyer conjecture (see Section 2 for the statement and background). If $d > 0$ is squarefree with $\gcd(d, 3) = 1$, then there is a non-trivial solution to $x^3 + y^3 = z^3$ in $\mathbb{Q}(\sqrt{d})$ if and only if*

$$\begin{aligned} &\#\{(x, y, z) \in \mathbb{Z}^3 : x^2 + y^2 + 7z^2 + xz = d\} \\ &= \#\{(x, y, z) \in \mathbb{Z}^3 : x^2 + 2y^4 + 4z^2 + xy + yz = d\}. \end{aligned}$$

If $d > 0$ is squarefree with $3|d$, then there is a non-trivial solution to $x^3 + y^3 = z^3$ in $\mathbb{Q}(\sqrt{d})$ if and only if

$$\begin{aligned} &\#\{(x, y, z) \in \mathbb{Z}^3 : x^2 + 3y^2 + 27z^2 = d/3\} \\ &= \#\{(x, y, z) \in \mathbb{Z}^3 : 3x^2 + 4y^2 + 7z^2 - 2yz = d/3\}. \end{aligned}$$

Moreover, there are non-trivial solutions in $\mathbb{Q}(\sqrt{d})$ if and only if there are non-trivial solutions in $\mathbb{Q}(\sqrt{-3d})$.

Remark. *Only one direction of our result is conditional on the Birch and Swinnerton-Dyer conjecture. As mentioned in Section 2, it is known that if $E/\mathbb{Q}$ is an elliptic curve, $L(E,1) \neq 0$ implies that $E(\mathbb{Q})$ is finite. As a consequence, if the number of representations of d (respectively $d/3$) by the two different quadratic forms are different, then there are no solutions in $\mathbb{Q}(\sqrt{d})$.*

Our method is similar to that used by Tunnell [26] in his solution to the congruent number problem. The congruent number problem is to determine, given a positive integer n, whether there is a right triangle with rational side lengths and area n. It can be shown that n is a congruent number if and only if the elliptic curve $E_n : y^2 = x^3 - n^2 x$ has positive rank. The Birch and Swinnerton-Dyer states that E_n has positive rank if and only if $L(E_n, 1) \neq 0$, and Waldspurger's theorem (roughly speaking) states that

$$f(z) = \sum_{n=1}^{\infty} n^{1/4} \sqrt{L(E_n, 1)} q^n, \quad q = e^{2\pi i z}$$

is a weight $3/2$ modular form. Tunnell computes this modular form explicitly as a difference of two weight $3/2$ theta series and proves that (in the case that n is odd), E_n is congruent if and only if n has the same number of representations in the form $x^2 + 4y^2 + 8z^2$ with z even as it does with z odd. Tunnell's work was used in [14] to determine precisely which integers $n \leq 10^{12}$ are congruent (again assuming the Birch and Swinnerton-Dyer conjecture).

Remark. *In* [20]*, Soma Purkait computes two (different) weight* $3/2$ *modular forms whose coefficients interpolate the central critical L-values of twists of $x^3 + y^3 = z^3$ (see Proposition 8.7). Purkait expresses the first as a linear combiation of 7 theta series, but does not express the second in terms of theta series.*

An outline of the paper is as follows. In Section 2 we will discuss the Birch and Swinnerton-Dyer conjecture. In Section 3 we will develop the necessary background. This will be used in Section 4 to prove Theorem 1.

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2. Elliptic Curves and the Birch and Swinnerton-Dyer Conjecture

The smooth, projective curve $C : x^3 + y^3 = z^3$ is an elliptic curve. Specifically, if $X = \frac{12z}{y+x}$ and $Y = \frac{36(y-x)}{y+x}$, then

$$E : Y^2 = X^3 - 432.$$

From Euler's proof of the $n = 3$ case of Fermat's last theorem, it follows that the only rational points on $x^3 + y^3 = z^3$ are $(1 : 0 : 1)$, $(0 : 1 : 1)$, and $(1 : -1 : 0)$. These

correspond to the three-torsion points $(12, -36)$, $(12, 36)$, and the point at infinity on E.

Suppose that $K = \mathbb{Q}(\sqrt{d})$ is a quadratic field and $\sigma : K \to K$ is the automorphism given by $\sigma(a + b\sqrt{d}) = a - b\sqrt{d}$ with $a, b \in \mathbb{Q}$. If $P = (x, y) \in E(K)$, define $\sigma(P) = (\sigma(x), \sigma(y)) \in E(K)$. Then, $Q = P - \sigma(P) \in E(K)$ and $\sigma(Q) = -Q$. Since the inverse of $(x, y) \in E(K)$ is $(x, -y)$, it follows that $P - \sigma(P) = (a, b\sqrt{d})$ for $a, b \in \mathbb{Q}$. Thus, (a, b) is a rational point on the quadratic twist E_d of E, given by

$$E_d : dY^2 = X^3 - 432.$$

Lemma 2. *The point (a, b) on $E_d(\mathbb{Q})$ is in the torsion subgroup of $E_d(\mathbb{Q})$ if and only if the corresponding solution to $x^3 + y^3 = z^3$ is trivial.*

This lemma will be proven in Section 4. Thus, there is a non-trivial solution in $\mathbb{Q}(\sqrt{d})$ if and only if $E_d(\mathbb{Q})$ has positive rank.

If $E/\mathbb{Q}$ is an elliptic curve, let

$$L(E, s) = \sum_{n=1}^{\infty} \frac{a_n(E)}{n^s}$$

be its L-function (see [24], Appendix C, Section 16 for the precise definition). It is known (see [6]) that $L(E, s) = L(f, s)$ for some weight 2 modular form $f \in S_2(\Gamma_0(N))$, where N is the conductor of E. It follows from this that $L(E, s)$ has an analytic continuation and functional equation of the form

$$\Lambda(E, s) = (2\pi)^{-s} N^{s/2} \Gamma(s) L(E, s)$$

and $\Lambda(E, s) = w_E \Lambda(E, 2 - s)$, where $w_E = \pm 1$ is the root number of E. Note that if $w_E = -1$, then $L(E, 1) = 0$. The weak Birch and Swinnerton-Dyer conjecture predicts that

$$\mathrm{ord}_{s=1} L(E, s) = \mathrm{rank}(E(\mathbb{Q})).$$

The strong form predicts that

$$\lim_{s \to 1} \frac{L(E, s)}{(s-1)^r} = \frac{\Omega(E) R(E/\mathbb{Q}) \prod_p c_p \# Ш(E/\mathbb{Q})}{(\# E_{\mathrm{tors}})^2}.$$

Here, $\Omega(E)$ is the real period of E times the number of connected components of $E(\mathbb{R})$, $R(E/\mathbb{Q})$ is the elliptic regulator, the c_p are the Tamagawa numbers, and $Ш(E/\mathbb{Q})$ is the Shafarevich-Tate group.

Much is known about the Birch and Swinnerton-Dyer in the case when $\mathrm{ord}_{s=1} L(E, s)$ is 0 or 1. See for example [9], [13], [17], and [22]. The best known result currently is the following.

Theorem 3 (Gross-Zagier, Kolyvagin, et. al.)**.** *Suppose that $E/\mathbb{Q}$ is an elliptic curve and $\mathrm{ord}_{s=1} L(E, s) = 0$ or 1. Then, $\mathrm{ord}_{s=1} L(E, s) = \mathrm{rank}(E(\mathbb{Q}))$.*

The work of Bump-Friedberg-Hoffstein [7] or Murty-Murty [18] is necessary to remove a condition imposed in the work of Gross-Zagier and Kolyvagin.

3. Preliminaries

If d is an integer, let χ_d denote the unique primitive Dirichlet character with the property that

$$\chi_d(p) = \left(\frac{d}{p}\right)$$

for all odd primes p. This character will be denoted by $\chi_d(n) = \left(\frac{d}{n}\right)$, even when n is not prime.

If λ is a positive integer, let $M_{2\lambda}(\Gamma_0(N), \chi)$ denote the $\mathbb{C}$-vector space of modular forms of weight 2λ for $\Gamma_0(N)$ with character χ, and $S_{2\lambda}(\Gamma_0(N), \chi)$ denote the subspace of cusp forms. Similarly, if λ is a positive integer, let $M_{\lambda+\frac{1}{2}}(\Gamma_0(4N), \chi)$ denote the vector space of modular forms of weight $\lambda + \frac{1}{2}$ on $\Gamma_0(4N)$ with character χ and $S_{\lambda+\frac{1}{2}}(\Gamma_0(4N), \chi)$ denote the subspace of cusp forms. We will frequently use the following theorem of Sturm [25] to prove that two modular forms are equal.

Theorem 4. *Suppose that $f(z) \in M_r(\Gamma_0(N), \chi)$ is a modular form of integer or half-integer weight with $f(z) = \sum_{n=0}^{\infty} a(n)q^n$. If $a(n) = 0$ for $n \leq \frac{r}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]$, then $f(z) = 0$.*

We denote by T_p the usual index p Hecke operator on $M_{2\lambda}(\Gamma_0(N), \chi)$, and by T_{p^2} the usual index p^2 Hecke operator on $M_{\lambda+1/2}(\Gamma_0(4N), \chi)$.

Next, we recall the Shimura correspondence.

Theorem 5 ([23])**.** *Suppose that $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_{\lambda+1/2}(\Gamma_0(4N), \chi)$. For each squarefree integer t, let*

$$\mathcal{S}_t(f(z)) = \sum_{n=1}^{\infty} \left(\sum_{d \mid n} \chi(d) \left(\frac{(-1)^{\lambda} t}{d}\right) d^{\lambda-1} a(t(n/d)^2) \right) q^n.$$

Then, $\mathcal{S}_t(f(z)) \in M_{2\lambda}(\Gamma_0(2N), \chi^2)$.

One can show using the definition that if p is a prime and $p \nmid 4tN$, then

$$\mathcal{S}_t(f|T_{p^2}) = S_t(f)|T_p,$$

that is, the Shimura correspondence commutes with the Hecke action.

In [27], Waldspurger relates the Fourier coefficients of a half-integer weight Hecke eigenform f with the central critical L-values of the twists of the corresponding integer weight modular form g with the same Hecke eigenvalues. To state it, let $\mathbb{Q}_p$ be the usual field of p-adic numbers. Also, if

$$F(z) = \sum_{n=1}^{\infty} A(n)q^n,$$

let $(F \otimes \chi)(z) = \sum_{n=1}^{\infty} A(n)\chi(n)q^n$.

Theorem 6 ([27], Corollaire 2, p. 379)**.** *Suppose that $f \in S_{\lambda+1/2}(\Gamma_0(4N), \chi)$ is a half-integer weight modular form and $f|T_{p^2} = \lambda(p)f$ for all $p \nmid 4N$. Denote the Fourier expansion of $f(z)$ by*

$$f(z) = \sum_{n=1}^{\infty} a(n)q^n, \quad q = e^{2\pi i z}.$$

If $F(z) \in S_{2\lambda}(\Gamma_0(2N), \chi^2)$ is an integer weight modular form with $F(z)|T_p = \lambda(p)g$ for all $p \nmid 4N$ and n_1 and n_2 are two squarefree positive integers with $n_1/n_2 \in \left(\mathbb{Q}_p^\times\right)^2$ for all $p|N$, then

$$a(n_1)^2 L(F \otimes \chi^{-1}\chi_{n_2\cdot(-1)^\lambda}, \lambda)\chi(n_2/n_1)n_2^{\lambda-1/2} = a(n_2)^2 L(F \otimes \chi^{-1}\chi_{n_1\cdot(-1)^\lambda}, \lambda)n_1^{\lambda-1/2}.$$

Our goal is to construct two modular forms $f_1(z) \in S_{3/2}(\Gamma_0(108))$ and $f_2(z) \in S_{3/2}(\Gamma_0(108), \chi_3)$ that have the same Hecke eigenvalues as

$$F(z) = q\prod_{n=1}^{\infty}(1-q^{3n})^2(1-q^{9n})^2 \in S_2(\Gamma_0(27)).$$

This is the weight 2 modular form corresponding to $E_1 : y^2 = x^3 - 432$. As in [26], we will express f_1 and f_2 as linear combinations of ternary theta functions. The next result recalls the modularity of the theta series of positive-definite quadratic forms.

Theorem 7 (Theorem 10.9 of [15])**.** *Let A be a $r \times r$ positive-definite symmetric matrix with integer entries and even diagonal entries. Let $Q(\vec{x}) = \frac{1}{2}\vec{x}^T A\vec{x}$, and let*

$$\theta_Q(z) = \sum_{n=0}^{\infty} r_Q(n)q^n$$

be the generating function for the number of representations of n by Q. Then,

$$\theta_Q(z) \in M_{r/2}(\Gamma_0(N), \chi_{\det(2A)}),$$

where N is the smallest positive integer so that NA^{-1} has integer entries and even diagonal entries.

Finally, we require some facts about the root numbers of the curves E_d. If $F(z) \in S_2(\Gamma_0(N))$ is the modular form corresponding to E, let $F(z)|W(N) = N^{-1}z^{-2}f\left(-\frac{1}{Nz}\right)$. Then $F(z)|W(N) = -w_E F(z)$ (see for example Theorem 7.2 of [15]). Theorem 7.5 of [15] states that if ψ is a quadratic Dirichlet character with conductor r and $\gcd(r, N) = 1$, then $F \otimes \psi \in S_2(\Gamma_0(Nr^2))$ and

$$(F \otimes \psi)|W(Nr^2) = (\psi(N)\tau(\psi)^2/r)F|W(N)$$

where $\tau(\psi) = \sum_{m=1}^{r} \psi(m)e^{2\pi i m/r}$ is the usual Gauss sum.

Suppose d is an integer so that $|d|$ is the conductor of χ_d and $F(z) \in S_2(\Gamma_0(27))$ is the modular form corresponding to E_1. Then $F \otimes \chi_d$ is the modular form corresponding to E_d. Using the result from the previous paragraph and the equality $\tau(\chi_d)^2 = |d|\chi_d(-1)$, we get

$$w_{E_d} = w_{E_1}\chi_d(27)\chi_d(-1) = \chi_d(-27).$$

provided $\gcd(d, 3) = 1$.

4. Proofs

In this section, we prove Lemma 2 and Theorem 1.

Before we prove Lemma 2, we will first need to determine the order of torsion subgroup of $E_d(\mathbb{Q})$. First, note that $6\sqrt{2} \notin \mathbb{Z}$ and hence $E_d(\mathbb{Q})$ has no element of order two. Since there are no elements of order 2 in $E_d(\mathbb{Q})_{\text{tors}}$, then $2 \nmid |E_d(\mathbb{Q})_{\text{tors}}|$. We will now show $q \nmid E_d(\mathbb{Q})_{\text{tors}}$ for primes $q > 3$.

If p is prime with $p \equiv 2 \pmod 3$, then we have that the map $x \to x^3 \in \mathbb{F}_p$ is a bijection. Since this is a bijection, we have that $\sum_{x=0}^{p-1}\left(\frac{f(x)}{p}\right) = 0$. Thus we have $\#E(\mathbb{F}_p) = p + 1$. Suppose that $|E_d(\mathbb{Q})_{\text{tors}}| = N$ for N odd.

If we suppose that a prime $q > 3$ divides N then we can find an integer x that is relatively prime to $3q$ so that $x \equiv 2 \pmod 3$ and $x \equiv 1 \pmod q$. By Dirichlet's Theorem, we have an infinite number of primes contained in the arithmetic progression $3nq + x$ for $n \in \mathbb{N}$. If we take p to be a sufficiently large prime in this progression, then the reduction of $E_d(\mathbb{Q})_{\text{tors}} \subseteq E(\mathbb{F}_p)$ has order N. So, now we have $q||E_d(\mathbb{F}_p)| = p+1 \equiv x+1 \equiv 2 \pmod q$. This is a contradiction. Hence the only prime that divides N is 3. We can follow a similar argument to show that 9 does not divide N. This means that the torsion subgroup of $E_d(\mathbb{Q})$ is either $\mathbb{Z}/3\mathbb{Z}$ or trivial.

Futhermore, if $E_d(\mathbb{Q})$ contains a point of order 3 then the $x-$coordinate of the point must be a root of the three-division polynomial $\phi_3(x) = 3x^4 - 12(432)d^3x$. The only real roots to $\phi_3(x)$ are $x = 0$ and $x = 12d$. For $x = 0$, then we have $y = \pm 108$ and $d = -3$. Finally for $x = 12d$, then we find that $y = 1296d^3$ and that $d = 1$. Thus we conclude that $E_d(\mathbb{Q}) \cong \mathbb{Z}/3\mathbb{Z}$ if and only if $d \in \{1, -3\}$. Finally, the torsion subgroup of $E_d(\mathbb{Q})$ is trivial for $d \notin \{1, -3\}$.

Proof of Lemma 2. ($\Rightarrow$) Let $(x, y) \in E_d(\mathbb{Q})$ so that (x, y) is not in $E_d(\mathbb{Q})_{\text{tors}}$. By doing some arithmetic we get that $(x, y\sqrt{d}) \in E(K)$. In Section 2, we defined a map from $C(K) \to E(K)$. The inverse of this map sends

$$(x, y\sqrt{d}) \to \left(\frac{\frac{12}{x} + \frac{y\sqrt{d}}{3x}}{2}, \frac{\frac{12}{x} - \frac{y\sqrt{d}}{3x}}{2}\right) \in C(K).$$

If we suppose that this is a trivial solution to C, then either the $x-$coordinate or $y-$coordinate is zero. Hence $y = \pm\frac{36\sqrt{d}}{d}$.

If $d = 1$, then we have $y = -36$ and $x = 12$. From Section 2, we know that $(12, -36)$ corresponds to $(1 : 0 : 1)$ which is a trivial solution to C. Hence the point (x, y) does not satisfy the hypothesis for $d = 1$. Now for $d \neq 1$, we have $y \notin \mathbb{Q}$. This contradicts the hypothesis for $d \neq 1$. Hence the solution we have is non-trivial.

($\Leftarrow$) Let (x, y, z) be a non-trivial solution to $x^3 + y^3 = z^3$ in K. Note that for $d = 1$ or -3 Euler showed that there are only trivial solutions and thus this direction is vacuously true for these two cases.

For $d \neq 1$ and -3, from Section 2 we showed that $(x, y, z) \to (X, Y) = P \in E(K)$. Also from section 2, if $P - \sigma(P) = (a, b\sqrt{d})$ then $(a, b) \in E_d(\mathbb{Q})$. Since $d \neq 1$ and -3, then the torsion subgroup of $E_d(\mathbb{Q})$ is trivial. Thus $(a, b) \notin E_d(\mathbb{Q})_{\text{tors}}$. $\square$

Recall from Section 3 that the elliptic curve E corresponds to the modular form $F(z) = q\prod_{n=1}^{\infty}(1 - q^{3n})^2(1 - q^{9n})^2 \in S_2(\Gamma_0(27))$.

Remark. *For convenience we will think of $F(z)$ as a Fourier series with coefficients $\lambda(n)$ for $n \in \mathbb{N}$. Note that if $\lambda(n) \neq 0$ then $n \equiv 1 \pmod 3$. So we can write $\lambda(n) = \lambda(n)(\frac{n}{3})$ for $n \in \mathbb{N}$. Hence $F \otimes \chi_{-3d} = F \otimes \chi_d$. We can now conclude that $L(E_d, 1) = L(E_{-3d}, 1)$.*

Proof of Theorem 1. To begin we will examine the case for $d < 0$ so that $d \equiv 2 \pmod 3$. Note that $\dim S_{3/2}(\Gamma_0(108), \chi_1) = 5$. Moreover, we have the following basis of $S_{3/2}(\Gamma_0(108), \chi_1))$:

$$\begin{aligned}
g_1(z) &= q - q^{10} - q^{16} - q^{19} - q^{22} + 2q^{28} + \cdots, \\
g_2(z) &= q^2 - q^5 + q^8 - q^{11} + q^{14} - 2q^{17} - q^{20} + \cdots, \\
g_3(z) &= q^3 - 2q^{12} + \ldots, \\
g_4(z) &= q^4 - q^{10} + q^{13} - q^{16} - q^{19} - q^{22} - q^{25} + q^{28} + \cdots, \text{ and} \\
g_5(z) &= q^7 - q^{10} + q^{13} - q^{16} - q^{22} - q^{25} + \cdots.
\end{aligned}$$

By Theorem 4 we have:

$$\begin{aligned}
\mathcal{S}_1(g_1(z) + g_4(z)) &= F(z) + F(z)|V(2), \\
\mathcal{S}_2(g_1(z) + g_4(z)) &= 0, \\
\mathcal{S}_3(g_1(z) + g_4(z)) &= 0, \\
\mathcal{S}_1(g_1(z) + g_5(z)) &= F(z), \\
\mathcal{S}_2(g_1(z) + g_5(z)) &= 0, \text{ and} \\
\mathcal{S}_3(g_1(z) + g_5(z)) &= 0.
\end{aligned}$$

Since we took $t = 1, 2,$ and 3, then from Section 2 we have $\mathcal{S}_t((g_1(z) + g_4(z))|T_{p^2}) = \mathcal{S}_t((g_1(z) + g_4(z))|T(p)$ for all primes $p > 3$. Since $F(z)$ and $F(z)|V(2)$ are both Hecke eigenforms, then $F(z)|T(p) = \lambda(p)F(z)$ and $F(z)|V(2)|T(p) = \lambda(p)F(z)|V(2)$ for primes $p > 3$. Also, since 1, 2 and 3 divide $4N$, then we have $(g_1(z) + g_4(z)|T_{p^2} -$

$\lambda(p)(g_1(z)+g_4(z)$ is in: $\ker(\mathcal{S}_1)$, $\ker(\mathcal{S}_2)$, and $\ker(\mathcal{S}_3)$. Furthermore since $\ker(\mathcal{S}_1)\cap\ker(\mathcal{S}_2)\cap\ker(\mathcal{S}_3)=0$, then $g_1(z)+g_4(z)$ is a Hecke eigenform. The case of $g_1(z)+g_5(z)$ is similar.

We will now take the quadratic forms $Q_1(x,y,z)=x^2+3y^2+27z^2$ and $Q_2(x,y,z)=3x^2+4y^2-2yz+7z^2$. We have their theta-series $\theta_{Q_1},\theta_{Q_2}\in M_{3/2}(\Gamma_0(108),\chi_1)$. Also by Theorem 4, we have

$$\theta_{Q_1}(z)-\theta_{Q_2}(z)=-2(g_1(z)+g_5(z))+4(g_1(z)+g_4(z)).$$

Furthermore, since $g_1(z)+g_4(z)$ and $g_1(z)+g_5(z)$ are both Hecke eigenforms with the same eigenvalues then $\theta_{Q_1}(z)-\theta_{Q_2}(z)$ is a Hecke eigenform as well.

Let $a(n)$ denote the $n-$th cofficient of $\theta_{Q_1}(z)-\theta_{Q_2}(z)$. By Theorem 6, we have

$$L(E_{-n_2},1)=\sqrt{\frac{n_2}{n_1}}\left(\frac{a(n_2)}{a(n_1)}\right)^2L(E_{-n_1},1)$$

for n_1 and n_2 squarefree with $\left(\frac{n_1/n_2}{p}\right)=1$ for $p=3$ and $n_1/n_2\equiv 1\pmod 8$. If we take $n_2\equiv 1\pmod 3$, then the table below covers all possible cases.

n_2	n_1	$a(n_1)$	$L(E_{-n_1},1)$
$n_2\equiv 1\pmod{24}$	1	2	$1.52995\dots$
$n_2\equiv 34\pmod{48}$	34	4	$1.04953\dots$
$n_2\equiv 19\pmod{24}$	19	-6	$0.70199\dots$
$n_2\equiv 13\pmod{24}$	13	2	$0.42434\dots$
$n_2\equiv 22\pmod{48}$	22	-4	$1.30474\dots$
$n_2\equiv 7\pmod{24}$	7	-2	$1.15653\dots$
$n_2\equiv 10\pmod{36}$	10	-4	$1.93525\dots$
$n_2\equiv 46\pmod{48}$	46	4	$0.90231\dots$

Thus we have $d<0$ with $d\equiv 2\pmod 3$ and $L(E_d,1)=0$ if and only if $a(-d)=0$. Since $L(E_{3n_2},1)=L(E_{-n_2},1)$ then we have $d>0$ so that $d\equiv 3\pmod 9$ and $L(E_d,1)=0$ if and only if $a(d/3)=0$.

We will now examine the case $d<0$ so that $3|d$ and $d\equiv 6\pmod 9$. Note that $\dim S_{3/2}(\Gamma_0(108),\chi_3)=5$, and we have the basis:

$$\begin{aligned}
h_1(z)&=q-2q^{13}-q^{25}-2q^{28}+\cdots,\\
h_2(z)&=q^2+q^5-q^8-q^{11}-q^{14}-q^{20}-2q^{23}+2q^{26}+\cdots,\\
h_3(z)&=q^4-q^{13}-2q^{16}+2q^{25}-q^{28}+\cdots,\\
h_4(z)&=q^7-q^{13}-q^{19}+\cdots,\text{ and}\\
h_5(z)&=q^{10}-q^{16}-q^{19}-q^{22}+q^{25}+\cdots.
\end{aligned}$$

By Theorem 4, we have

$$\begin{aligned}
\mathcal{S}_1(h_1(z) - h_4(z) + 2h_5(z)) &= F(z),\\
\mathcal{S}_2(h_1(z) - h_4(z) + 2h_5(z)) &= 0,\\
\mathcal{S}_3(h_1(z) - h_4(z) + 2h_5(z)) &= 0,\\
\mathcal{S}_1(h_1(z) - 4h_3(z) - 5h_4(z) + 10h_5(z)) &= F(z) + 4F(z)|V(2),\\
\mathcal{S}_2(h_1(z) - 4h_3(z) - 5h_4(z) + 10h_5(z)) &= 0, \text{ and}\\
\mathcal{S}_3(h_1(z) - 4h_3(z) - 5h_4(z) + 10h_5(z)) &= 0.
\end{aligned}$$

From a similar argument as the previous case, we get that $h_1(z) - h_4(z) + 2h_5(z)$ and $h_1(z) - 4h_3(z) - 5h_4(z) + 10h_5(z)$ are Hecke eigenforms for $T(p^2)$ for primes $p > 3$. We will now take the quadratic forms $Q_3(x, y, z) = x^2 + y^2 + 7z^2 + xz$ and $Q_4(x, y, z) = x^2 + 2y^2 + 4z^2 + xy + yz$.

We will denote the theta series corresponding to Q_3 and Q_4 by θ_{Q_3} and θ_{Q_4}, respectively. Note that $\theta_{Q_3}, \theta_{Q_4} \in M_{3/2}(\Gamma_0(108), \chi_3)$. By Theorem 4, $\theta_{Q_3} - \theta_{Q_4} = 2h_1(z) - 4h_3(z) - 6h_4(z) + 12h_5(z)$. Since $h_1(z) - h_4(z) + 2h_5(z)$ and $h_1(z) - 4h_3(z) - 5h_4(z) + 10h_5(z)$ have the same eigenvalues, $\theta_{Q_3} - \theta_{Q_4}$ is a Hecke eigenform. Let $b(n)$ denote the $n-$th coefficient of $\theta_{Q_3} - \theta_{Q_4}$.

Hence by Theorem 6, we have

$$L(E_{-3n_2}, 1) = \sqrt{\frac{n_2}{n_1}} \left(\frac{b(n_2)}{b(n_1)}\right)^2 L(E_{-3n_1}, 1).$$

n_2	n_1	$b(n_1)$	$L(E_{-3n_1}, 1)$
$n_2 \equiv 1 \pmod{24}$	1	2	$0.58887\ldots$
$n_2 \equiv 34 \pmod{48}$	34	12	$1.81785\ldots$
$n_2 \equiv 19 \pmod{24}$	19	-6	$0.60794\ldots$
$n_2 \equiv 13 \pmod{24}$	13	6	$1.46993\ldots$
$n_2 \equiv 22 \pmod{48}$	22	-12	$2.25989\ldots$
$n_2 \equiv 7 \pmod{24}$	7	-6	$1.00159\ldots$
$n_2 \equiv 10 \pmod{36}$	10	12	$3.35196\ldots$
$n_2 \equiv 46 \pmod{48}$	46	-12	$1.56286\ldots$

Therefore if $d < 0$ and $d \equiv 6 \pmod 9$, $L(E_d, 1) = 0$ if and only if $b(-d/3) = 0$. Furthermore by the remark, we have

$$L(E_{n_2}, 1) = \sqrt{\frac{n_2}{n_1}} \left(\frac{b(n_2)}{b(n_1)}\right)^2 L(E_{-3n_1}, 1)$$

for $n_2 \equiv 1 \pmod 3$. Thus for squarefree $d > 0$ so that $d \equiv 1 \pmod 3$, $L(E_d, 1) = 0$ if and only if $b(d) = 0$.

There are two pairs of cases that Theorem 6 does not handle: $d > 0$ with $d \equiv 6 \pmod 9$ and $d < 0$ with $d \equiv 1 \pmod 3$, and $d > 0$ with $d \equiv 2 \pmod 3$ and $d < 0$ with $d \equiv 3 \pmod 9$.

We will now consider $d > 0$ with $d \equiv 2 \pmod 3$. To handle this case, we will show that the root number of E_d is -1. Recall from the end of Section 3 that $w_{E_d} = \chi_d(-27)$. Hence $w_{E_d} = \chi_d(-27) = \left(\frac{d}{-27}\right) = -1$. Furthermore, by the remark we have $w_{E_d} = -1$ for $d < 0$ so that $d \equiv 3 \pmod 9$.

We now want to show that there are no non-trivial solutions for $d > 0$ with $d \equiv 6 \pmod 9$. To do this, we will show that $x^2 + 3y^2 + 27z^2 = 3x^2 + 4y^2 - 2yz + 7y^2 \neq d$. Since $d > 0$ then $-d/3 \equiv 1 \pmod 3$. This means that $x^2 + 3y^2 + 27z^2 \equiv 0$ or $1 \pmod 3$. We also have that $3x^2 + 4y^2 - 2yz + 7z^2 \equiv (y + 2z)^2 \equiv 0$ or $1 \pmod 3$. Hence $r_{Q_1}(d) = r_{Q_2}(d) = 0$.

Let ψ be the non-trivial Dirichlet character with modulus 3. Note that $(\theta_{Q_3}(z) - \theta_{Q_4}) \otimes \psi \in M_{3/2}(\Gamma_0(108 * 3^2), \chi_3\psi^2)$ by Proposition 3.12 in [19]. By Theorem 4, $(\theta_{Q_3}(z) - \theta_{Q_4}) \otimes \psi = \theta_{Q_3}(z) - \theta_{Q_4}(z)$. So $b(n)\psi(n) = b(n)$ for all $n \geq 1$. If $d \equiv 2 \pmod 3$, we have that $\psi(d) = -1$. Thus $b(d) = -b(d)$. Therefore $b(d) = 0$. Thus $r_{Q_3}(d) = r_{Q_4}(d)$ for $d \equiv 2 \pmod 3$.

Hence we have shown that by checking the number of solutions of the pair of equations $x^2 + y^2 + 7z^2 + xz$ and $x^2 + 2y^2 + 4z^2 + xy + yz$, and $x^2 + 3y^2 + 27z^2$ and $3x^2 + 4y^2 + 7z^2 - 2yz$ is sufficient to determine when there are non-trivial solutions to $x^3 + y^3 = z^3$ in $\mathbb{Q}(\sqrt{d})$. □

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